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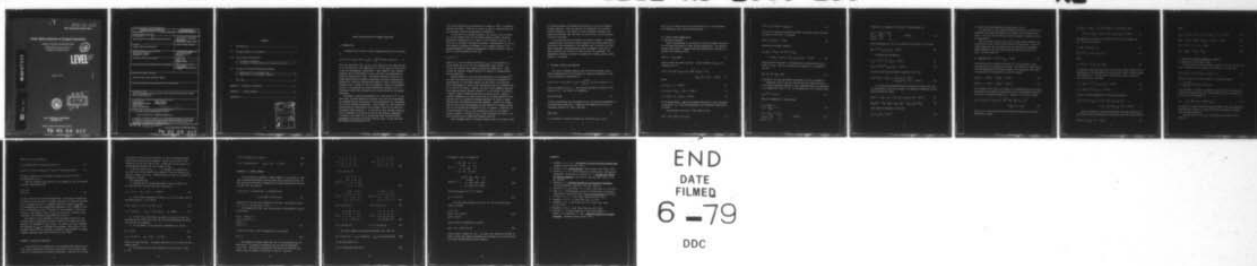
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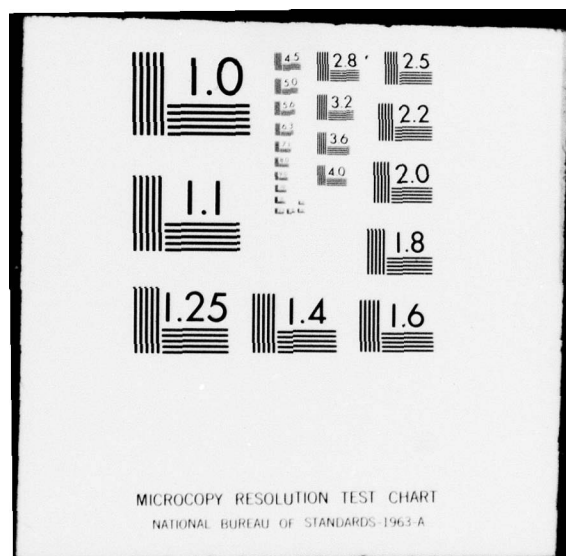
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Cubic Spline Solution of Integral Equations

GEORGE P. MUELLER AND MERVINE ROSEN

*Radiation-Matter Interactions Branch
Radiation Technology Division*

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The solution of linear integral equations of the form $f_1'(x)g'(x) + f_2(x)g(x) + f_3(x) = \int_{a(x)}^{b(x)} dy K(x,y)g(P(x,y))$ is considered. The equation is reduced to a set of linear algebraic equations in the values of g(x) on a finite mesh. The method entails formal orthogonal polynomial quadrature of the integral and subsequent formal cubic spline interpolation for the values of g at the mesh points. Discontinuities in the derivatives of g are accommodated.		

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CONTENTS

I.	Introduction.....	1
II.	Integral Equation and Notation.....	3
III.	Cubic Spline Interpolation	
	A. The Spline Equations.....	4
	B. Inclusion of Discontinuities in Derivatives of $g(x)$	7
IV.	Solving the Integro-Differential Equation	
	A. Evaluation of the Integral Term.....	9
	B. Matrix Form of the Integral Equation.....	10
V.	Test Case.....	11
	Appendix A. Auxiliary Conditions.....	12
	Appendix B. A Simple Example.....	14
	References.....	17

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CUBIC SPLINE SOLUTION OF INTEGRAL EQUATIONS

I. INTRODUCTION

A general form in which a linear integral equation can be written is

$$f_d(x) g'(x) + f_{li}(x) g(x) + f_{in}(x) = \int_a^b dy K(x,y) g(P(x,y)) , \quad (1)$$

where the functions $f_d(x)$, $f_{li}(x)$, $f_{in}(x)$ and $K(x,y)$ are known and $g(x)$ is to be solved for. The subscripts d, li and in stand for derivative, linear and inhomogenous, respectively. Most commonly, such integral equations are not susceptible to exact solution; some approximate or numerical method must be used. This report presents a version of a simple method based on numerical quadrature and spline interpolation. Essentially, the spline interpolation method is used to functionally represent $g(x)$ in terms of a set of parameters. Given this parameterization, we can perform the various operations indicated in Eq. (1), obtaining a simple matrix equation that can be solved for the unknown parameters.

The method of spline interpolation of a function $g(x)$,¹⁻⁴ in its usual form, involves breaking an interval $[a,b]$ into $M-1$ subintervals and representing the function $g(x)$ on each subinterval by a polynomial (we use cubic splines exclusively). The M points $\{x_i\}$ that define the subintervals are called knots; the M values of $g(x)$ at the knots are customarily known ($g_i = g(x_i)$). In the case of cubic splines, there are $4(M-1)$ parameters to be set-- M of them by the values g_i themselves. In addition, the values of the splines and their first and second derivatives are matched at the $M-2$ interior knots. We are then left

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with $4(M-1)-[M+3(M-2)]=2$ conditions still needed in order to completely specify the system. There are a variety of ways of choosing these last two conditions; several are discussed in Appendix A.

In the case of an integral equation, the values g_i are not known, but we can formally carry out the spline analysis anyway, obtaining splines that represent $g(x)$ in terms of the M unknown values g_i . We then have a functional (piecewise polynomial) representation of $g(x)$ and can proceed to evaluate the integral in the integral equation by an appropriate method of quadrature--generally a numerical method. What is left is a matrix equation of the form

$$U_{i,j} g_j = V_i, \quad (2)$$

where the g_j are the unknown quantities, $U_{i,j}$ are the elements of a square matrix and the V_i are the inhomogeneous terms from Eq. (1). (We use the notation that repeated indices are summed over.) Thus, solving the original integral equation is reduced to straightforward matrix operations.

Methods similar to this have been described in the literature⁵⁻¹¹ and tests and qualifications on their use have been investigated.^{6,11} Unfortunately, we find these results too particular to the specific cases investigated to predict from them whether there are instabilities in our use of the method. Instead, we adopt a more pragmatic approach. Given some particular integral equation (1) to be solved, we construct a similar integral equation whose exact solution we know. Based on intuitive arguments, one often has an idea of the gross behavior of the solution $g(x)$ of a particular integral equation. We choose a simple function $g(x)$ that exhibits that gross behavior. Further, we choose a form for $K(x,y)$ that approximates the real kernel but is simple enough that we can perform the integration in Eq. (1) exactly. Similarly, we choose functions f_d and f_{11} that approximate those in the original integral equation. We then set f_{in} so that the given trial $g(x)$ is the exact solution of the constructed integral equation.

This stratagem yields an integral equation that can be used to test

the various numerical procedures that we want to use on the original integral equation. Since we tailored the properties of the exactly solvable equation to the original, we are reasonably confident that numerical procedures that work well for the one will not give spurious values for the other.

In the next sections we describe our method in detail. Included is a generalization of the usual spline approach that allows known discontinuities in the derivatives of the solution $g(x)$ to be built into the spline equations. Following that we give an example of the sort of model equations that we solve in order to test our procedures. For pedagogical purposes, we present in Appendix B a simple, exactly solved, example of our method.

II. INTEGRAL EQUATION AND NOTATION

The type of integral equation that we will solve using a cubic spline representation of the solution and numerical quadrature to perform the necessary integrations is

$$f_d(x) g'(x) + f_{li}(x) g(x) + f_{in}(x) = \int_{y_1(x)}^{y_2(x)} dy K(x,y) g(P(x,y)), \quad (3)$$

which is similar to Eq. (1). The various functions are defined on the interval $[a,b]$. The limits of integration,

$$a \leq y_1(x) < y_2(x) \leq b, \quad (4)$$

could be incorporated into the kernel $K(x,y)$, but are more convenient in this explicitly stated form. Similarly, we presume that $P(x,y)$ is a continuous function of x and y such that

$$a \leq P(x,y) \leq b. \quad (5)$$

For the moment, we further presume that functions $f_d(x)$, $f_{li}(x)$

and $f_{in}(x)$, as well as $P(x,y)$ and the kernel $K(x,y)$, are continuous with continuous first and second derivatives.

III. CUBIC SPLINE INTERPOLATION

A. The Spline Equations

Before concentrating on the problem of solving integral equations, let us review the method of cubic spline interpolation. Our notation follows that of Ref. 4. Suppose that some function $g(x)$ is known at M points on the interval $[a,b]$,

$$a=x_1 < x_2 < \dots < x_{M-1} < x_M = b. \quad (6)$$

These M points are known as knots. In each interval $[x_i, x_{i+1}]$, we represent $g(x)$ by

$$g_i(x) = g_i s_i(x) + g_{i+1} t_i(x) - \frac{h_i^2}{6} [g''_i (s_i - s_i^3) + g''_{i+1} (t_i - t_i^3)], \quad i=1, M-1 \quad (7)$$

where

$$h_i = x_{i+1} - x_i, \quad i=1, M-1, \quad (8)$$

$$s_i = s_i(x) = (x_{i+1} - x)/h_i, \quad i=1, M-1, \quad (9)$$

$$t_i = t_i(x) = (x - x_i)/h_i, \quad i=1, M-1. \quad (10)$$

The M values $g''_1, g''_2, \dots, g''_M$ are the second derivatives of $g(x)$ evaluated at the knots x_i ; they are determined by the continuity conditions at the knots.

In particular, the form (7) was chosen so that

$$g(x_i) = g_i = g_i(x_i) = g_{i-1}(x_i) \quad (11)$$

$$g''(x_i) = g''_i = g''_i(x_i) = g''_{i-1}(x_i) ; \quad (12)$$

that is, the values and second derivatives of adjacent cubic functions are equal at each knot. By setting

$$g'_i(x_i) = g'_{i-1}(x_i) , \quad i=1, M-1 \quad (13)$$

we obtain the further condition

$$\begin{aligned} h_{i-1} g''_{i-1} + 2(h_{i-1} + h_i) g''_i + h_i g''_{i+1} \\ = 6 [(g_{i+1} - g_i)/h_i - (g_i - g_{i-1})/h_{i-1}] , \quad i=2, M-1 \end{aligned} \quad (14)$$

which gives $M-2$ equations for determining the M unknown values g''_i in terms of the g_i . Two more conditions are needed; for the moment we will use

$$g''_1 = g''_2 \quad \text{and} \quad g''_{M-1} = g''_M . \quad (15)$$

A discussion of these and similar conditions is given in Appendix A.

Henceforth, we will use the notation that each index runs from 1 to M , unless otherwise indicated. Also, repeated indices are understood to be summed over, from 1 to M . We can express Eq. (14) in matrix form

$$L_{j,i} g''_i = 6 R_{j,i} g_i \quad (16)$$

where all elements of L vanish except

$$\begin{aligned} L_{1,1} &= L_{M,M} = 1 \\ L_{1,2} &= L_{M,M-1} = -1 \end{aligned} \quad (17)$$

$$\left. \begin{aligned} L_{i,i} &= 2(h_{i-1} + h_i) \\ L_{i,i-1} &= h_{i-1} \\ L_{i,i+1} &= h_i \end{aligned} \right\} \quad i=2, M-1. \quad (18)$$

Similarly, all elements of the matrix R vanish except for

$$\left. \begin{aligned} R_{i,i} &= -1/h_{i-1} - 1/h_i \\ R_{i,i-1} &= 1/h_{i-1} \\ R_{i,i+1} &= 1/h_i \end{aligned} \right\} \quad i=2, M-1. \quad (19)$$

Upon multiplying Eq. (16) by the inverse of the matrix L, we obtain

$$g_i'' = 6 (L^{-1})_{i,j} R_{j,k} g_k, \quad i=1, M-1. \quad (20)$$

We define two new matrices

$$A_{i,k} = -h_i^2 (L^{-1})_{i,j} R_{j,k}, \quad i=1, M-1 \quad (21)$$

$$B_{i,k} = -h_i^2 (L^{-1})_{i+1,j} R_{j,k}, \quad i=1, M-1 \quad (22)$$

so that we can write the spline functions in the form

$$g_i(x) = g_i s_i + g_{i+1} t_i + A_{i,k} g_k (s_i - s_i^3) + B_{i,k} g_k (t_i - t_i^3), \quad i=1, M-1. \quad (23)$$

In solving the integro-differential equation, we shall also need the derivatives of $g(x)$ at the knots. By taking the derivative of each spline function we obtain

$$g_i'(x_i) = [g_{i+1} - g_i + 2 A_{i,k} g_k + B_{i,k} g_k] / h_i, \quad i=1, M-1 \quad (24)$$

$$g_{M-1}'(x_M) = [g_M - g_{M-1} - A_{M-1,k} g_k - 2 B_{M-1,k} g_k] / h_{M-1}, \quad (25)$$

which could be rewritten in the form

$$g_i' = D_{i,k} g_k, \quad i=1, M-1. \quad (26)$$

B. Inclusion of Discontinuities in Derivatives of $g(x)$

The solutions of the integro-differential equations in which we are interested often have discontinuities in their first or second derivatives. The cubic spline method is readily adapted to account for such discontinuities.

When we create a mesh of knots for the cubic spline procedure, we naturally set a knot at each of the points of discontinuity. For notational simplicity, we will suppose that the function $g(x)$ possesses discontinuities in its first and second derivatives at each of its knots. We write these as

$$G'_i = \lim_{y \rightarrow 0} \frac{d}{dx} (g(x+y) - g(x-y))|_{x=x_i}, \quad i=2, M-1 \quad (27)$$

and similarly for the second derivative G''_i . For the moment, these discontinuities are presumed to be known. The addition of these discontinuities generates the following changes in our equations: The continuity equations for derivatives (12) and (13) become

$$g''_{i-1}(x_i) + G''_i(x_i) = g''_i(x_i), \quad i=2, M-1 \quad (28)$$

$$g'_{i-1}(x_i) + G'_i(x_i) = g'_i(x_i), \quad i=2, M-1. \quad (29)$$

The constants, such as g''_i , that appear in the expressions for the spline functions are the values of derivatives of $g(x)$ as determined from below the knots; for example, $g''_i = g''_{i-1}(x_i)$. Thus Eq. (7) for $g_i(x)$ becomes

$$g_i(x) = g_i s_i(x) + g_{i+1} t_i(x) - \frac{h_i^2}{6} [(g''_i + G''_i)(s_i - s_i^3) + g''_{i+1}(t_i - t_i^3)] \quad (30)$$

and Eq. (14), which arises from the continuity equation for the first derivatives, becomes

$$h_{i-1} g_{i-1}'' + 2 (h_{i-1} + h_i) g_i'' + h_i g_{i+1}'' = -2 h_i G_i'' - h_{i-1} G_{i-1}'' - 6 G_i' + 6 [(g_{i+1} - g_i)/h_i - (g_i - g_{i-1})/h_{i-1}], \quad i=2, M-1. \quad (31)$$

The definitions (17-19) of the matrices L and R do not change, but Eqs. (16) and (20) become

$$L_{j,i} g_i'' = 6 R_{j,i} g_i + r_j \quad (32)$$

$$g_i'' = (L^{-1})_{i,j} [6 R_{j,k} g_k + r_j], \quad (33)$$

where

$$r_i = -6 G_i' - 2 h_i G_i'' - h_{i-1} G_{i-1}'' . \quad (34)$$

If auxiliary conditions other than (15) are used, slight changes in the definitions of both the matrices L and R and the vector r will be needed (see Appendix A). With the use of the matrices A and B, defined in Eqs. (21) and (22), and with

$$X_i = -\frac{1}{6} h_i^2 [(L^{-1})_{i,k} r_k + G_i''] , \quad i=1, M-1 \quad (35)$$

$$Y_i = -\frac{1}{6} h_i^2 (L^{-1})_{i+1,k} r_k , \quad i=1, M-1 \quad (36)$$

we have our final expression for the spline functions,

$$g_i(x) = g_i s_i + g_{i+1} t_i + (s_i - s_i^3) [A_{i,k} g_k + X_i] + (t_i - t_i^3) [B_{i,k} g_k + Y_i] , \quad i=1, M-1 . \quad (37)$$

With the addition of the discontinuities, the first derivative of the spline functions evaluated at (just above) each knot can be written

$$g_i'(x_i) = D_{i,k} g_k + d_i , \quad i=1, M-1 \quad (38)$$

where

$$D_{i,k} = [\delta_{k,i+1} - \delta_{k,i} + 2 A_{i,k} + B_{i,k}] / h_i, \quad i=1, M-1 \quad (39)$$

$$D_{M,k} = [\delta_{M,k} - \delta_{k,M-1} - A_{M-1,k} - 2 B_{M-1,k}] / h_{M-1} \quad (40)$$

$$d_i = [2 X_i + Y_i] / h_i, \quad i=1, M-1 \quad (41)$$

$$d_M = [-X_{M-1} - 2 Y_{M-1}] / h_{M-1}. \quad (42)$$

IV. SOLVING THE INTEGRO-DIFFERENTIAL EQUATION

A. Evaluation of the Integral Term

Let us rewrite our basic integral equation, Eq. (3), with the variable x taking on only the values of the knots x_i . We then have

$$f_d(x_i) g(x_i) + f_{li}(x_i) g(x_i) + f_{in}(x_i) = I_i, \quad (43)$$

where for each of the M knots there is a different integral

$$I_i = \int_{y_1(x_i)}^{y_2(x_i)} dy K(x_i, y) g(P(x_i, y)) \quad (44)$$

to be evaluated. Further, we choose for each value of x_i a numerical quadrature procedure, with associated weights $w_{i,k}$ and ordinates $y_{i,k}$, so the I_i can be expressed as

$$I_i = \sum_{k=1}^N w_{i,k} K(x_i, y_{i,k}) g(P(x_i, y_{i,k})). \quad (45)$$

We suppose, in order to keep the notation simple, that each quadrature uses N points; actually, an entirely different quadrature procedure could be used for each x_i .

For each value of $P(x_i, y_{i,k})$ we determine the quantity $p(i,k)$ such that

$$x_{p(i,k)} \leq P(x_i, y_{i,k}) \leq x_{p(i,k)+1}, \quad (46)$$

so that we can replace $g(P(x_i, y_{i,k}))$ by the appropriate spline function $g_{p(i,k)}(P(x_i, y_{i,k}))$.

We rewrite I_i in the form

$$I_i = I_{i,m} g_m + J_i, \quad (47)$$

where

$$J_i = w_{i,k} K(x_i, y_{i,k}) [X_p (s_p - s_p^3) + Y_p (t_p - t_p^3)], \quad (48)$$

$$I_{i,j} = w_{i,k} K(x_i, y_{i,k}) [\delta_{p,j} s_p + \delta_{p+1,j} t_p + A_{p,j} (s_p - s_p^3) + B_{p,j} (t_p - t_p^3)]. \quad (49)$$

For given i and k , $p=p(i,k)$ is determined by Eq. (46); s_p and t_p are defined by

$$s_p = (x_{p+1} - P(x_i, y_{i,k})) / h_p, \quad (50)$$

$$t_p = (P(x_i, y_{i,k}) - x_p) / h_p. \quad (51)$$

B. Matrix Form of the Integral Equation

We can now collect the terms generated in the last few sections and rewrite the integral equation in the form

$$U_{i,m} g_m = V_i, \quad (52)$$

which has the solution

$$g_m = (U^{-1})_{m,i} V_i. \quad (53)$$

The quantities U and V in Eq. (52) are

$$U_{i,m} = I_{i,m} - f_{li}(x_i) \quad i,m - f_d(x_i) D_{i,m} \quad (54)$$

$$V_i = -J_i + f_{in}(x_i) + f_d(x_i) d_i \quad (55)$$

V. TEST CASE

As a numerical test of our method, we use an integral equation that models a radiation damage process. Specifically, as a heavy ion traverses an amorphous material, some of its energy is lost to elastic, displacement producing, collisions. The following integral equation yields the energy $g(x)$ that the ion with initial energy x gives up to such elastic collisions.⁸⁻¹⁰ The integral equation has the same form as Eq. (3); specifically,

$$f_d(x) g'(x) + f_{li}(x) g(x) + f_{in}(x) = \int_1^x dy x^{\frac{1}{2}} y^{-\frac{3}{2}} g(y) U(y-1) + \int_1^{x-1} dy x^{\frac{1}{2}} y^{-\frac{3}{2}} g(x-y) U(y-1) \quad (56)$$

with the added condition that $g(0)=0$. The various functions in Eq. (56) are given in the following table; $U(y)$ is the unit Heavyside function.

Functional Forms Appearing in Equation (56)

Function	$0 < x < 1$	$1 < x < 2$	$2 < x$
f_d	$kx^2 + 2x^{\frac{1}{2}}$	$kx^2 + 2x^{\frac{1}{2}}$	$kx^2 + 2x^{\frac{1}{2}}$
f_{li}	0	$2(x^{\frac{1}{2}} - 1)$	$2(x^{\frac{1}{2}} - 1)$
f_{in}	$-kx^2 - 2x^{\frac{1}{2}}$	$f_1(x)$	$f_1(x) + f_2(x)$

where f_1 and f_2 are given by

$$f_1(x) = -2kx^2 + 0.5kx^{\frac{3}{2}} - x^{\frac{1}{2}} \ln x - 4x^{\frac{3}{2}} + 10(x - x^{\frac{1}{2}}) + 1 \quad (57)$$

$$f_2(x) = 2x^{\frac{1}{2}} [2(x+1) + (4-5x)(x-1)^{-\frac{1}{2}} - \sin^{-1} x^{-\frac{1}{2}} + \sin^{-1}((x-1)/x)^{\frac{1}{2}}] \quad (58)$$

The exact solution of this integral equation is $g(x)=x$ for $0 < x < 1$ and $g(x)=2x-x^{\frac{1}{2}}$ for $x > 1$.

With the condition that $g(0)=0$, we can examine Eq. (54) for discontinuities, discovering that

$$\begin{aligned} G'(1) &= 1/2 \\ G''(1) &= 1/4 \end{aligned} \quad (59)$$

and that there are no other discontinuities. The computer program that we wrote to solve this sort of integral equation has a grid of 48 values of x_1 , ranging from $x=0$ to a somewhat arbitrarily chosen $x=x_{\max}=400$. The integrations are done using Gauss quadrature, with a change of variable to compensate for the $y^{-\frac{3}{2}}$ behavior of the integrand. (More particularly, the interval from 0 to x_{\max} is broken into a number of sub-intervals, with different quadratures used in each.)

The result of this procedure is that the spline solution of the integral equation reproduces the exact results to no worse than 0.3%--generally the agreement is much better. Of course, the error increases and decreases as x_{\max} is increased and decreased. The quality of this agreement leads us to be confident in using the same methods for similar integral equations whose exact solutions are not known.

APPENDIX A: AUXILIARY CONDITIONS

As we noted in the introduction, the straightforward application of cubic spline interpolation requires the setting of two auxiliary conditions to be satisfied by the spline coefficients. Usually, one of these

is imposed at each end of the interval, so that the various matrices remain balanced about their diagonals. Below, we list five of the conditions that we have used at $x=x_1$, noting that for each there is a corresponding condition that can be used at $x=x_M$.

We use the notation $H_{1,i} = [\quad]$ to indicate the components of the first row of the H matrix; any elements in that row that are not listed are taken to be zero. Similar notation is used to indicate the elements of a vector.

The conditions are:

a) Suppose that the second derivative of $g(x)$ is known at x_1 , so that $g''(x_1)=g^*$, then the auxiliary condition is

$$L_{1,i} = [1,] , \quad R_{1,i} = [0,] , \quad r_i = [g^*,] . \quad (A1)$$

b) If the first derivative is known at x_1 , say $g'(x_1)=g^*$, then by differentiating Eq. (7) we obtain

$$6 (g_2 - g_1)/h_1 - 6 g^* = 2 h_1 g_1'' + h_1 g_2'' \quad (A2)$$

$$L_{1,i} = [2h_1, h_1,] , \quad R_{1,i} = [-1/h_1, 1/h_1,] , \quad r_i = [-6g^*,] . \quad (A3)$$

c) If both $g'(x_1)$ and $g''(x_1)$ are known, we can use (a) and (b), provided that Eq. (30) is not used for $i=2$ and provided that G_2' and G_2'' are included, if necessary.

d) In the absence of any particular information, we can set

$$g_1'' = a^* g_2'' , \quad (A4)$$

$$L_{1,i} = [1, -a^*,] , \quad R_{1,i} = [0,] , \quad r_i = [0,] , \quad (A5)$$

where a^* is some constant. We usually take $a^*=1$, but $1/2$ and 0 are also commonly used.⁴

e) We can also set the third derivatives of $g(x)$ equal at $x=x_2$, so that

$$h_2 g_1'' - (h_1 + h_2) g_2'' + h_1 g_3'' = 0, \quad (A6)$$

$$L_{1,i} = [h_2, -h_1 - h_2, h_1,] , \quad R_{1,i} = [0,], \quad r_i = [0,] . \quad (A7)$$

APPENDIX B: A SIMPLE EXAMPLE

In this section we present a simple example of our method, by solving an integro-differential equation whose exact solution we know. We will outline the solution by supplying the various matrices and vectors. We start with the equation

$$\begin{aligned} -(x-2) g'(x) + (x^2 - 3x + 8) g(x) + (-2x^3 + 6x^2 - 14x - 4) \\ = 3(x-2) \int_2^x dy U(y-2) g(y) \end{aligned} \quad (B1)$$

where $U(y)$ is the unit step (Heavyside) function. We choose the knots $x_i = 1, 2, 3, 4$, so that $h_i = 1$, $i=1, 3$.

If we examine Eq. (B1), and, in particular, differentiate it twice, we find that

$$\begin{aligned} g(2^-) &= g(2^+) = 4 \\ g'(2^-) &= g'(2^+) = 2 \\ g''(2^-) &= 0 \\ g''(2^+) &= 6, \end{aligned} \quad (B2)$$

so that all of the G' and G'' quantities are zero except

$$G''_2 = 6. \quad (B3)$$

This example is simple enough that all of the evaluations can be done exactly, including the integrals over the spline polynomials in Eq. (B1). We thus can successively form various the matrices and vectors that are defined in Sections III and IV. They are:

$$L = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix}$$

$$R = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad (B4)$$

$$r = [0, -12, -6, 0]$$

$$24 L^{-1} = \begin{vmatrix} 19 & 5 & -1 & 1 \\ -5 & 5 & -1 & 1 \\ 1 & -1 & 5 & -5 \\ 1 & -1 & 5 & 19 \end{vmatrix} \quad (B5)$$

$$-24 A = \begin{vmatrix} 5 & -11 & 7 & -1 \\ 5 & -11 & 7 & -1 \\ -1 & 7 & -11 & 5 \\ -- & -- & -- & -- \end{vmatrix} \quad -24 B = \begin{vmatrix} 5 & -11 & 7 & -1 \\ -1 & 7 & -11 & 5 \\ -1 & 7 & -11 & 5 \\ -- & -- & -- & -- \end{vmatrix} \quad (B6)$$

$$8 X = [3, -5, 1, --] \quad 8 Y = [3, 1, 1, --] \quad (B7)$$

$$8 D = \begin{vmatrix} -13 & 19 & -7 & 1 \\ -3 & -3 & 7 & -1 \\ 1 & -7 & 3 & 3 \\ -1 & 7 & -19 & 13 \end{vmatrix} \quad 16 I = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 26 & 26 & -2 \\ -2 & 38 & 122 & 34 \end{vmatrix} \quad (B8)$$

$$8 d = [9, -9, 3, -3] \quad 16 J = [0, 0, -6, -6] \quad (B9)$$

The vector forms of the various functions in Eq. (B1) are

$$f_d = [1, 0, -1, -2] \quad f_{11} = [6, 6, 8, 12] \quad f_{1n} = [-14, -24, -46, -92] \quad (B10)$$

so that the vector V is

$$16 V = [-206, -384, -736, -1454]. \quad (B11)$$

The matrix U and its inverse are

$$16 U = \begin{vmatrix} -70 & -38 & 14 & -2 \\ 0 & -96 & 0 & 0 \\ 0 & 12 & -96 & 4 \\ -6 & 66 & 46 & -106 \end{vmatrix} \quad (B12)$$

$$16374 U^{-1} = \begin{vmatrix} -3747 & 1453 & -522 & 51 \\ 0 & -2729 & 0 & 0 \\ 9 & -423 & -2778 & -105 \\ 216 & -1965 & -1176 & -2520 \end{vmatrix} \quad (B13)$$

Upon multiplying V by U^{-1} , we obtain

$$g_1 = [2, 4, 9, 20] \quad (B14)$$

If we use these values of g_i in Eq. (37) for the spline functions, we obtain

$$\begin{aligned} g_1(x) &= 2x \\ g_2(x) &= 2x + 3(x-2)^2 \\ g_3(x) &= g_2(x) \end{aligned} \quad (B15)$$

Thus we exactly reproduce the function

$$g(x) = 2x + 3(x-2)^2 U(x-2) \quad (B16)$$

that we used to create Eq. (B1). Of course, the reason the solution is exact is that only simple polynomials are involved, but we see the ease with which discontinuities can be handled.

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